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Using the state space to record the behavioural effects of symmetry in the Tower of Hanoi problem and an isomorph

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The state space of a problem has offered a powerful technique for representing problems and describing the behaviour of problem solving subjects. In this study the effects of symmetry within the structure of a problem on the behaviour of subjects solving that problem is presented through three experiments. The first two experiments are fairly direct tests of symmetry effects in the Tower of Hanoi problem and an isomorph. The third experiment provides an indirect test of the use of symmetry by a problem task with no obvious symmetry, but containing symmetric subproblem tasks. The discussion focuses on the results of the symmetry tests as well as on the state space method for characterizing symmetry effects.

Introduction

The purpose of this paper is twofold. Firstly to present the state space representation of a problem situation and to demonstrate its utility for describing the symmetry decompositions of a problem. Secondly, in three experiments the state space method will be used to demonstrate the effect on problem solving behaviour of symmetries present in the structure of certain problems, namely the 4-ring Tower of Hanoi problem (TOH) and an isomorph, i.e. a problem differing in appearance, but identically structured.

The state space analysis of a problem is taken from mechanical problem solving theory. It results from the study of search through problem situations in an attempt to find efficient solutions. This approach to problem solving has been described by Amarel (1968) and Nilsson (1971). The state space method of analysing human problem solving behaviour has been described in research by Hayes & Simon (1975), Goldin & Luger (1975), and Luger (1976). In the Luger research the state space analysis was used to measure "goal directedness" and the use of "subgoals" by problem solving subjects. Further, the state space method has been used to measure transfer effects of subjects solving two problems of related structure (Simon & Hayes, 1976; Reed, Ernst & Banerji, 1974; Luger & Bauer, 1978; Luger, 1979).

The TOH and its state space will be used to exemplify this approach. Furthermore the effects on subjects' behaviour of the symmetry present in the TOH and an isomorph will be analysed in three experiments.

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In the TOH problem four concentric rings (labelled 1, 2, 3, 4, respectively) are placed on the first of three pegs (labelled A, B, C, respectively). Whenever the rings are placed on any peg the size order must be preserved, that is, larger rings must always be below smaller rings. The apparatus is pictured in Fig. 1(a). The object of the TOH problem is to transfer all the rings from peg A to peg C in the minimum number of moves. Only one ring may be moved at a time and, as noted above, no larger ring may be placed over a smaller ring on any peg.

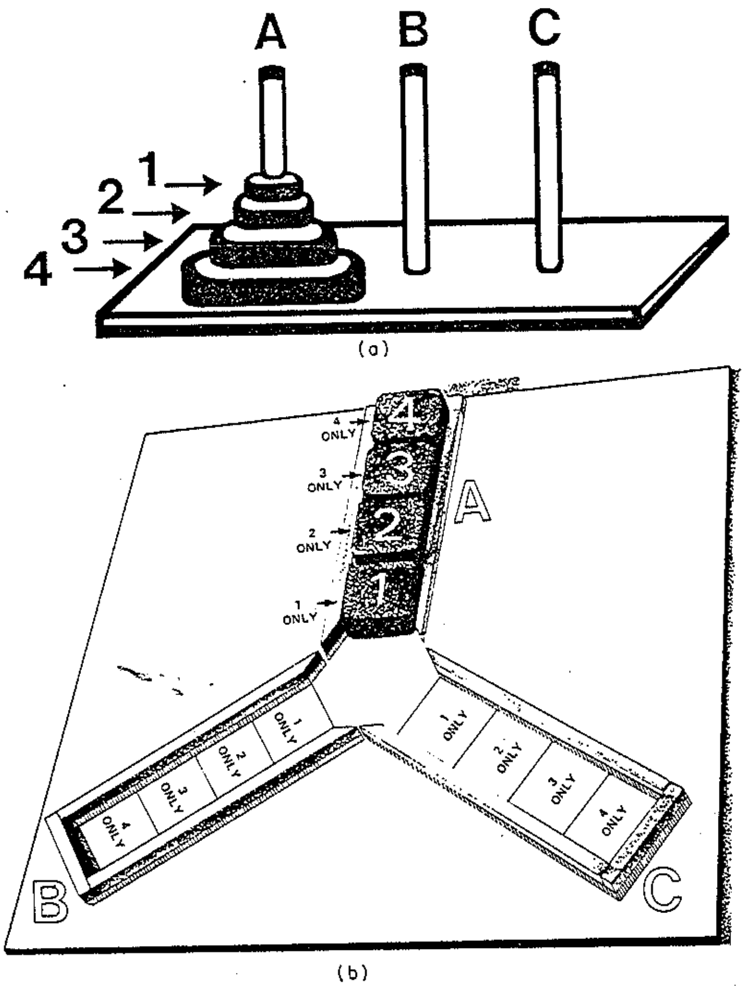

FIG. 1. (a) The Tower of Hanoi problem in its start position. A, B, C, 1, 2, 3, 4, relate to the state space of Fig. 2 and illustrate the isomorphism relationship with RRS. (b) The Railroad Switching problem in its start state. 1, 2, 3, 4 are replaced by engine, passenger car, mail car, guard car, respectively.

Figure 2 is the complete state space representation of the 4-ring TOH problem. The four letters labelling a state refer to the pegs on which the four rings are located. Thus, state CCBA indicates that ring 1 (the first "C" and smallest ring) and ring 2 (the second "C" and second smallest ring) are in their proper order on peg C; ring 3 (the "B" and

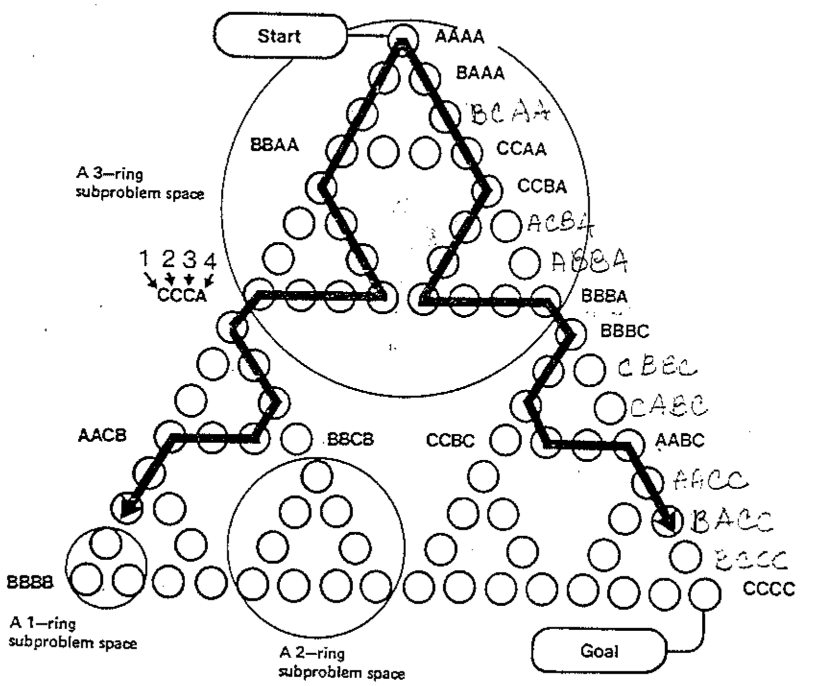

FIG. 2. The state space representation of the Tower of Hanoi and Railroad Switching problems. The four letters labelling a state refer to the pegs (tracks) on which the four rings (cars) are located. Legal moves effect transitions between adjacent states. Note examples of 1-, 2-, and 3-ring subproblem spaces and two paths symmetric in the state space.

second largest ring) is on peg B; while the largest ring is on peg A. The states adjacent in the diagram are connected by the legal moves of the problem. The solution path containing the minimum number of moves from peg A to peg C consists of the 15 steps from AAAA to CCCC down the right side of the state space diagram.

The TOH problem has a natural decomposition into nested subproblems. For example, to solve the 4-ring TOH it is necessary at some point to move the largest ring from peg A to either peg B or peg C. But before this can be done the three smaller rings must be assembled in their proper order on the other or "rest" peg, i.e. to move the largest ring from A to B, the three smaller rings must be on peg C. The problem of moving the top three rings from one peg to another is called a 3-ring subproblem of the 4-ring TOH. Thus the 4-ring TOH contains three 3-ring subspaces, differing by the position of ring 4. Similarly, each 3-ring subspace contains three 2-ring subspaces for a total of nine in the 4-ring TOH. Each 2-ring subspace may be further decomposed into three 1-ring subspaces. (Note examples of subproblem spaces in Fig. 2.)

It should be realized that the *structure* of moves within each n-ring subproblem is identical for any fixed n. Thus, even though the "start" and the "goal" pegs and the position of rings larger than n may differ in each instance, a one-to-one onto mapping exists which preserves the sets of possible legal moves within each subproblem. That is, each move within one subproblem corresponds with one and only one move within a second subproblem, and conversely. In this sense all TOH subproblems of n rings, for fixed n, are said to be isomorphic.

The TOH problem possesses considerable symmetry. For example, if the start state has all rings on peg A as in Fig. 2 any sequence of moves, or path within the TOH state space, has a symmetric opposite (called the symmetric *conjugate*) found by holding A fixed and exchanging B and C wherever they occur in the label of a state entered. This symmetry takes state BBAA into CCAA, state CCBC into state BBCB and the goal state CCCC into BBBB. (Note the example of two symmetric paths in Fig. 2.) Were the three pegs of the TOH board to be arranged at the corners of an equilateral triangle (as are the tracks of the Railroad Switching problem described below) the symmetry transformation would represent the geometric operation of reflection about an altitude of the triangle.

Similarly, each n-ring subproblem of the 4-ring TOH possesses symmetry mappings. In the top 3-ring subproblem of Fig. 2, for example, the two paths are symmetric. The paths through the bottom two 3-ring subproblems, however, are not symmetric in a 3-ring subproblem. They are examples of conjugate, i.e. identical paths through two (different) isomorphic 3-ring subproblems. Pairs of symmetry paths may be generated by taking one path, holding the letter representing the start peg constant, and exchanging the other two letters wherever they occur in the label of a state. An alternative description of symmetry of a TOH move sequence is to retain the "start" peg and exchange the roles of the "rest" and "goal" pegs wherever they occur in the sequence. For example, if the original move sequence took a set of rings from peg A to peg C using peg B as the temporary or "rest" peg, the symmetric conjugate would take the rings from peg A to peg B using peg C as the temporary or "rest" peg.

When the subject solving the 4-ring TOH problem makes a sequence of moves of rings on the TOH problem board, this sequence is represented as a series of paths through the TOH state space representation. Figure 3 represents the behaviour paths of one subject asked to solve the 4-ring TOH. In this example the start was AAAA and the goal state CCCC. The subject was asked to solve the problem in the fewest possible number of moves. This she did on her third attempt (trial 3). Note also that the second trial is interrupted at state CCCA, and is followed by its symmetric conjugate (the first half of trial 3) through the top 3-ring subproblem. This subject had been instructed that she could interrupt any trial and start over when she either "got confused or saw a better way of solving the problem".

In each of the three experiments described below, two tasks were given to each (independent) group of subjects. The first task required the subject to make a certain sequence of moves. For example, in Experiment 1, the subjects were asked to make the minimum sequence of moves of four rings from peg A to peg C. The subjects were encouraged to experiment with the moves, starting over when they got confused or saw a "better way". Only when they had met the criterion of the first task was the second task given, which was to make another sequence of moves that in some way included symmetries of the first task sequence. For example, in the first experiment the subject upon successfully moving the four rings from peg A to peg C was asked to move the four rings from peg A to peg B in the minimum number of moves. It was tested whether the subject immediately produced the symmetric conjugate to the previous minimal path solution or whether the subject passed through a trial-and-error period to discover the minimal solution as had been done in performing the first task. If the subject used a sequence of trials to get the second minimal path solution then little was "transferred" from the first task. If on the other hand, and as this paper predicts, the minimal solution

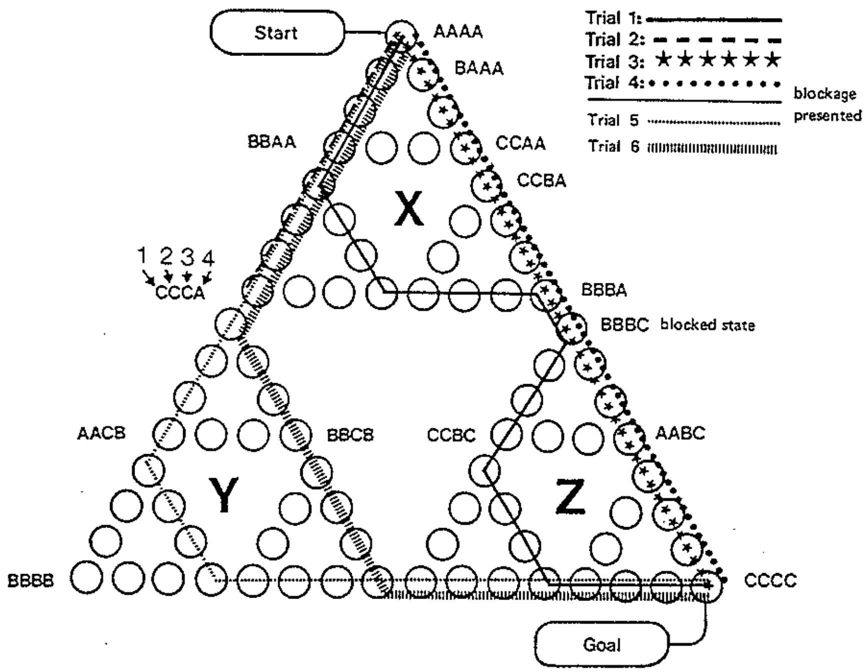

FIG. 3. The behaviour paths of one subject solving the TOH problem in Experiment 3. The first four trials make up the first task. State BBBC is the blocked state which the subject cannot enter in the second task. The 3-ring subproblems are labelled *X*, *Y*, and *Z*.

path is immediately produced, then: (1) the symmetries of a "learned" solution strongly effect the behaviour of a problem solving subject, (2) the state space representation of the problem situation can characterize the subject's symmetry behaviour in these instances, and (3) any strategies (Simon, 1975) used by subjects must have representation that includes possibility of generalization to the symmetries of a learned situation. This point will be addressed further in the discussion.

Finally, before describing the experiments, it is necessary to note the statistical test employed to measure the significance of the symmetry effect in the population of subjects. First, on examining subjects' paths immediately preceeding and following the test for symmetry in all three experiments, it was determined that 84% (237 of 282) of all paths through 3-ring subproblems, no matter where these subproblems occurred in the solution paths, were in the minimum number of moves. Secondly, there were two choices to be made in solving the top 3-ring subproblem: to start at state AAAA and go to the left to state CCCA, or to go to the right to state BBBA (cf. Fig. 2). Thus, the null hypothesis states that the probability of a symmetric path through the top 3-ring subproblem is the product of the probability of a minimum 3-ring subproblem solution and the probability of one of two equally likely 3-ring solutions or $(0{\cdot}84)\times(0{\cdot}5)$ or 42%. This expected value was compared with the actual problem solving data by the chi-square test.

Similarly, the expectation of a symmetric path through the 4-ring TOH problem was taken as the square of the probability of a symmetric path through a 3-ring subproblem.

This is because the 4-ring symmetry path is composed of two 3-ring paths. Again, this expected value was compared with the actual data using the chi-square test.

Experiment 1

METHOD

Thirty-two second-year Edinburgh University psychology undergraduates (15 male and 17 female) volunteered for the study. They were seated in a small well-lighted testing booth with the TOH board in the "start" state [Fig. 1(a)] in front of them. The experimenter, present throughout the problem solving session, recorded on a tape recorder the sequences of moves. This tape was later examined to reconstruct on the TOH state space (Fig. 3) the sequences of moves made during the problem solving session.

The instructions, read by the experimenter and available to the problem solver, explained the TOH problem, the "start" and "goal" states, and the set of legal moves: "only one ring may be moved at a time and no larger ring may be placed over a smaller ring on any peg". The subjects were told they could "start over any time they were confused or saw a better way of solving the problem". The first task subjects were given was to produce the minimal set of moves that took all four rings from peg A to peg C. If the subject took the rings to peg C but not in the minimal number of moves, he or she was informed that this was the case and encouraged to "start over and try to solve the problem in fewer moves".

When the 15 move solution criterion was met, the following instructions for the second task were given "Good, you have moved the rings from peg A to peg C in the fewest number of moves. Now, try to move the four rings from peg A to peg B in the fewest number of moves. Again you may start over if you get lost or see a better way of solving the problem". As mentioned in the preceeding section the second task was the symmetric conjugate of the first task.

RESULTS AND DISCUSSION

The results of Experiment 1 are given in Table 1. Of the 32 Ss, 21 produced perfect symmetry solutions for the first 3-ring subproblem encountered in the second task. That is, for the first 3-ring subproblem, the first trial of the second task was the symmetric conjugate of the last trial of the first task for 21 of 32 subjects. Using the expected values discussed in the previous section and the chi-square test this is significant at the 0·01 level.

Of the 32 Ss, 16 produced a perfect symmetry solution through the entire 4-ring TOH state space. That is, the entire first trial of the second task was the symmetric conjugate of the last trial of the first task. Using the chi-square test and the expected values discussed in the previous section this is significant at the 0·001 level.

It is interesting to note on further analysis of the data for the 32 Ss of Experiment 1, that six of the 11 Ss not showing perfect symmetry in their first trial of the second task actually started this trial towards goal CCCC (the goal of the previous task). Before reaching this goal, however, the subjects interrupted their paths and started towards goal BBBB (the symmetric conjugate). All six Ss completed the symmetric path through the top 3-ring subproblem and four of the six completed the entire symmetry to goal

TABLE 1
Data and significance levels for Experiments 1 *and* 2

	Exp. 1 ($N = 32$)		Exp. 2 ($N = 25$)	
	Number of Ss	Significance	Number of Ss	Significance
Perfect 3-ring symmetry	21	0·01	17	0·02
Symmetric interruption followed by 3-ring symmetry	6		4	
No symmetry	5		4	
Perfect 4-ring symmetry	16†	0·001	16‡	0·001
Symmetric interruption followed by 4-ring symmetry	4		3	
No symmetry	12		6	

† One subject took one extra step and immediately retracted it.
‡ Six subjects each added one extra step and immediately retracted it.

BBBB. Thus these subjects, although not meeting the strict symmetry criterion of the tasks, actually employed a symmetric pair of paths in attaining the desired goal.

Finally, during the first task of experiment one the Ss used an average of 3·7 trials before producing the minimum step criterion trial. There was an average of 19 moves in each of these non-minimal paths. The average time to production of the criterion path (not counting time between trials) was 340 s. Recall that the test for production of the symmetric opposite minimal step path was during the *first* trial of the second task.

In designing a second experiment it was decided to test for the presence of symmetry in subjects' solutions of an isomorph of the 4-ring TOH problem. In particular, an isomorph whose actual physical embodiment was in the form of an equilateral triangle in order that the problem symmetries might be more perceptually apparent. For this purpose the 4-car Railroad Switching problem was devised.

Experiment 2

In the Railroad Switching problem (RRS) a train of four cars (engine, passenger, mail, and guard cars) was parked in one of three tracks (labelled A, B, C) placed at the corners of an equilateral triangle. The object of the game was to transfer the four cars to another track with the restrictions that only one car may be moved at a time, and no car may be removed from the tracks. Labelled blocks make up the "train" and small constraints make up the "tracks". The device is pictured in Fig. 1(b). There are fixed stopping places for each car ("engine only", "passenger car only", etc.) marked on the tracks. This, along with the constraint that no car may be picked off the track corresponds to the TOH constraint that no larger ring may be placed over a smaller ring on any peg. In fact, when the task is given to move the cars from track A to track C, it may be checked that TOH and RRS are exact isomorphs. In particular, both are represented by the state space of Fig. 2, having the same "start" and "goal" states and the same symmetry and possible subproblem decompositions.

METHOD

Twenty-five second-year psychology students at the University of Edinburgh (seven males and 18 females) volunteered for the study. The testing situation and proceedures were exactly the same as for the first experiment except that the 4-car RRS was used in place of the 4-ring TOH. The subjects' first task was to move the four cars from track A to track C in the minimum number of moves. When this criterion was met on this task, the Ss were given the second task of moving the four cars from track A to track B in the fewest number of moves.

RESULTS AND DISCUSSION

The results of Experiment 2 are given in Table 1. Note that of 25 Ss, 17 produced, at the first trial of the second task, a symmetry path through the first 3-car subproblem. This path, as in the previous experiment, was the symmetric conjugate in that subproblem of the last trial of the first task. The chi-square test shows this to be significant at the 0·02 level.

For the entire 4-car RRS problem 16 Ss produced the symmetry path all the way to goal state BBBB on their first trial of the second task. Using the expected values discussed above and the chi square test this is significant at the 0·001 level.

Three of the eight Ss not having perfect symmetry paths through the top 4-car subproblem actually began their first trial of the second task towards goal CCCC. Before reaching that goal they interrupted their path, asked to start again, and proceeded towards goal BBBB. One of the eight subjects actually started and stopped two trials before completing the symmetric conjugate in the top 3-car subproblem of the last trial of the first task. Three of these four subjects produced symmetric paths through the entire 4-car RRS problem. Thus, these subjects although not meeting the strict symmetry criterion in completing the second task of Experiment 2 actually produced pairs of symmetric paths in the process of arriving at the production of this symmetry path.

Thus the number of symmetry paths produced in Experiment 2 seemed to be roughly equivalent to the number produced in Experiment 1. If it were desired to further study the importance of the physical aspects of the problem it would be necessary to design further experiments, perhaps placing the pegs of the TOH at the corners of an equilateral triangle as are the tracks in the RRS problem.

Before producing the minimal path criterion of the first task of Experiment 2, Ss used an average of four 22 state attempts. The average time to criterion for all Ss, not counting time between trials, was 352 s. The symmetry test was made on the first trial of the second task.

Experiment 3

The previous two experiments asked subjects to perform a second task that was the direct symmetric conjugate of the first task. It was decided in the third experiment, using again the 4-ring TOH problem, to give a second task that was not the direct symmetric conjugate of the first task, but that on the 3-ring level included path sections that were symmetric to portions of the previous task. Thus the third experiment was designed to study transfer effects not between consecutive minimal path solutions, as had Experiments 1 and 2, but tested for transfer to a new task that was longer than the first (by eight states) and although it was itself a minimal path, different portions of the task were symmetric to portions of the previous criterion path. Thus the criterion path for the second task contained two 3-ring subproblems directly symmetric to the 3-ring subproblems of the previous task and also contained a third 3-ring subproblem that was not symmetric to any portion of the criterion path of the first task. To design such an experiment the "blocked state" discussed in the next section was developed.

Finally, it was decided to set the criterion for the first task of Experiment 3 as two consecutive minimal path solutions. It was hoped that this would result in fewer interrupted symmetric paths at the beginning of the second task, as had occurred in the previous experiments (six and four Ss, respectively).

METHOD

Twenty-one fourth-year psychology students (nine male and 12 female) at the University of Edinburgh volunteered for this study. The TOH problem was used as in Experiment 1. The effects of symmetry were tested indirectly, however, using the following procedures: the first task given to the Ss was identical to the first task of Experiment 1 except that criterion was set as two consecutive minimal solution paths from peg A to peg C. If this criterion was met the subjects were given the second task. In this task they were told state BBBC would be "blocked" and could not be entered and asked to find a new minimal step solution from AAAA to CCCC, by-passing the blocked state.

Figure 3 gives the behaviour of a typical subject in the third experiment. The first four trials make up the solution to the first task, the last two trials making up the solution to the second task. It may be observed in Fig. 3 that the "blocked" state BBBC was a critical step in the 15 step minimal solution to the first task. In fact, this state, with the smallest three rings on peg B and the largest ring moving from peg A to peg C, provides the only direct access to the lower right-hand side 3-ring subproblem (marked *Z* in Fig. 3). With the blockage, this subproblem may only be entered from the opposite side (from 3-ring subproblem, *Y* in Fig. 3), with the three smallest rings on peg A and the movement of the largest ring from peg B to peg C. The minimum set of moves in the blocked situation then is 23. These are the moves of trial 6 in Fig. 3.

Note that the 23 move minimal solution is not symmetric on the 4-ring level with the first task of Experiment 3, but that paths through two of the three 3-ring subproblems that make up this solution are symmetric conjugates of paths through the 3-ring subproblems that make the criteria of the first task. These are the first eight moves through the top 3-ring subproblem (*X* in Fig. 3) and the last eight moves through the final 3-ring subproblem (*Z* in Fig. 3). Experiment 3 measured the effects of this symmetry on the behaviour of the problem solving subjects. In particular, the first attempt to solve the TOH problem with the blocked state in the minimum number of moves was examined to determine if the three 3-ring subproblems that make up the solution were solved in the minimum number of moves.

The instructions describing the blocking of state BBBC and the second task of finding the alternative minimal solution follow (a second board was used to demonstrate):

> Good, you have managed to solve the TOH correctly on two successive trials. In solving the problem you first moved rings 1, 2, and 3 on to peg B and then ring 4 on to

peg C. You then followed this by moving rings 1, 2, and 3 on to peg C. Before you is another TOH board with rings 1, 2, and 3 on B and ring 4 on C. This is one of the positions of the pieces on the direct path of the solution. This position now becomes illegal in that you cannot move the pieces into this position. This position thus represents a "blockage" in the direct solution path. I would like you again to solve the problem in a minimum number of moves but this time by-passing the blocked position. The new solution, of course, now consists of more moves than did the original solution.

The second TOH board, in the position of the blocked state, was placed in front of the problem solver but not so as to obstruct the TOH board the subject used.

RESULTS

The results of Experiment 3 are reported in Table 2. In the first 3-ring subproblem encountered in the second task 16 of 21 subjects produced the minimal solution. In the

TABLE 2

Data and significance levels in Experiment 3. *X, Y, and Z are the three* 3-*ring subproblems entered in task* 2 (*cf. Fig.* 3)

	Exp. 3 ($N=21$)	
	Perfect 3-ring subproblem symmetry	Significance
Subproblem X	16	0·01
Subproblem Y	7	NS
Subproblem Z	16	0·01

second 3-ring subproblem encountered seven of 21 subjects produced the minimal solution, and in the third 3-ring subproblem 16 of 21 subjects produced the minimal solution. Using the chi-square test and the criterion described above 16 of 21 minimal solutions is significant at the 0·01 level, seven of 21 minimal solutions is not significant and 16 of 21 minimal solutions is significant at the 0·01 level. Note again that the minimal paths through the first and third 3-ring subproblems are the symmetric conjugates of the 3-ring subproblems that make up the criterion trial of the first task of Experiment 3, while the minimal path through the second 3-ring subproblem is not the symmetric conjugate of any previous path the Ss have been asked to produce. Finally, the 21 Ss required an average of 3·9 trials, with an average of 20 states in each trial, before producing the criterion paths on the first task. The test for transfer effects was made on the first attempt of the second task.

Discussion

Symmetry present in a problem solving environment often plays an important role in the problem solvers' solutions (Polya, 1945; Gelernter, 1963). However true this statement may seem, there is very little evidence of *what* the role of symmetry is, *when* it is used by the problem solver or even *how* the symmetry present in a problem environment is to be characterized. This research has attempted to answer these questions in the limited domain of the TOH problem and an isomorph.

Firstly, the state space representation of a problem environment is proposed not only to accurately present all the steps or moves, goals, and blind alleys of the problem, but to characterize the "structure" of the problem including its subproblem and symmetry decompositions. Especially in problems like the Missionaries and Cannibals, Noughts and Crosses, NIM and the Tower of Hanoi, this method of problem representation may be quite powerful (Amarel, 1968; Goldin & Luger, 1975; Nilsson, 1971).

Secondly, the behaviour of subjects solving the problem may be described by paths through the state space representation of the problem. This is possible as long as the environmental situations considered distinct by the problem solver are represented by distinct states in the state space. Furthermore, the subproblems and symmetries used by the problem solver in the process of solution are readily available to empirical investigation.

Our research (Luger, 1976, 1979; Luger & Bauer, 1978) has proposed that problem solving is a process of discovering and using the invariants of the problem domain. These invariants may be known by researchers in an *a priori* analysis of the structure of the problem, where characterizations such as the state space used in this research may be helpful. The presence and the use of these invariants were bases for the hypotheses of this study. We proposed that as the subjects determined these problem invariants, they would be used in achieving the goals of the task. [Luger (1976, 1979) gives an account of subproblem invariance in the Tower of Hanoi problem and an isomorph.]

The symmetry invariant of the Tower of Hanoi problem domain is the main focus of this paper. The symmetry transformation of the Tower of Hanoi problem changes the values of certain observables of the problem such as "rest" and "target" pegs while preserving other relationships among the states of the problem, e.g. its nested sets of subproblems. The intent of this study was to give two tasks within each problem to each group of subjects. After meeting criteria on the first task, it was hypothesized that subjects would use a symmetry of the first task to perform the second task. That is to say, it was hypothesized that in their *first* attempt at the second task they would produce the symmetric conjugate of the solution to the first task (which had taken them about four trials to produce). The behaviour paths and the statistical analysis of these paths gives strong evidence that the symmetry invariants were in fact used by subjects in solving the second task in each of these experiments. The evidence is particularly compelling in the third experiment when Ss in the second task generated minimal path solutions through 3-ring subproblems when these were symmetric conjugates of 3-ring subproblem solutions of the first task and failed to generate minimal paths through the 3-ring subproblem that did not have a symmetric conjugate within the previous solution.

The spontaneous comments of subjects in the first two experiments lead further evidence to the use of symmetry. "It's a matter of changing everything around, isn't it?" "It should be the same, I suppose, if I just reverse the moves." "Both are exactly the same. You put it to the right and you put it to the left", and "you just have to get the intermediate items out of the way to get the largest to the goal". The use of general words such as "changing everything around", "both . . . the same", "reverse the moves", "largest", "intermediate items" and "goal" are strong indications that a symmetry transformation is utilized. But again, the strongest evidence for use of symmetry is not the words subjects use but the behaviour paths they create through the state space. These indeed form the statistical evidence for subjects' use of the symmetry invariants. Finally, it may be hypothesized that the problems' invariants make up "what is learned" in performing the first task in this study and offer an explanation of the transfer effects evident between the two tasks of each of the three experiments.

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